Shortest distance between a point and a function

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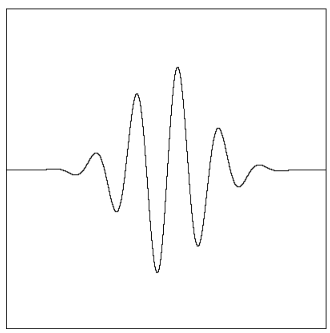
Credit

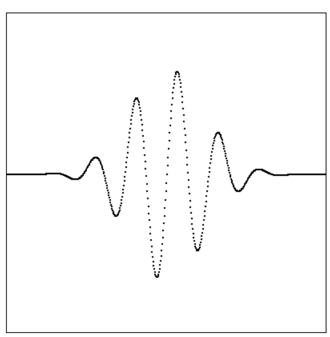
See the bottom of Iñigo Quilez' page at http://www.iquilezles.org/www/articles/distance/distance.htm.

The problem

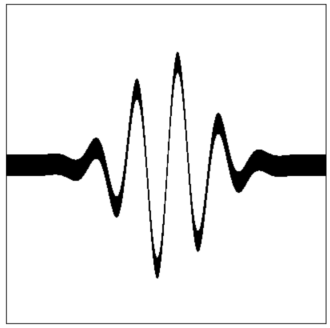
Suppose we have a function $f(x)$. As an example, we will use $f(x) = 2e^{-x^2}\sin(8x)$. This is plotted below over the range $x = -3 \ldots 3, y = -3 \ldots 3$.



If we try to draw this by cycling through successive x values and plotting the corresponding value of $f(x)$, we obtain a graph similar to this (x from -3 to $+3$, step 0.01):



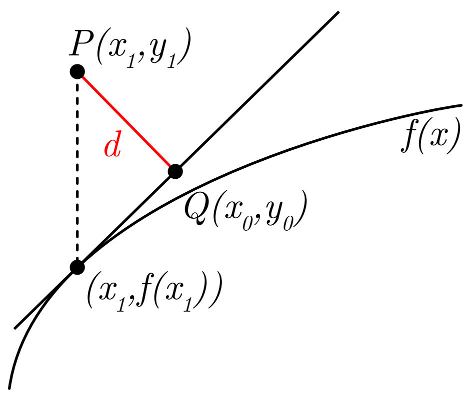
Gaps appear between the points whenever the gradient is steep. Obviously, in practice, we could draw short line segments *between* the points, giving an approximation to a smooth line, but we're trying to demonstrate something else here. Suppose we go through every point (i.e. pixel) in the range $x = -3 \ldots 3, y = -3 \ldots 3$. At each point, we work out the vertical distance from the point to the function, i.e. $|y - f(x)|$. If this is less than a certain limit, e.g. 0.2, colour the point black, otherwise colour it white. The result of this procedure is the following image.



It's now a continuous representation of the function, but it has some obvious problems, namely the variation in width, since we are only considering the *vertical* distance between the function and the chosen point. We need a means of finding the *closest* distance from the point to the function.

Derivation

Let our point be $P(x_1, y_1)$. As shown in the figure below, draw a tangent line to $f(x)$ at $x = x_1$. We will use the closest distance from $P(x_1, y_1)$ to the tangent line as an estimate of the closest distance between the point and the function. This distance is d, and the closest point on the tangent line is $Q(x_0, y_0)$. The estimate is reasonably valid when $P(x_1, y_1)$ is close to the function and the gradient of the function is not changing rapidly.



What is the equation of the tangent line? Its gradient is $f'(x_1)$ so its equation is

$$y = f'(x_1)\, x + b \tag{1}$$

To find b, remember the tangent line passes through the point $(x_1, f(x_1))$. Substituting this point in (1), we obtain

$$\begin{aligned} f(x_1) &= f'(x_1)\, x_1 + b \\ \Rightarrow b &= f(x_1) - f'(x_1)\, x_1 \end{aligned} \tag{2}$$

Substituting (2) back into (1), the full equation of the tangent is

$$y = f'(x_1)\, x + [f(x_1) - f'(x_1)\, x_1] \tag{3}$$

The shortest distance between a point (x_1, y_1) and a line $y = mx + b$ is

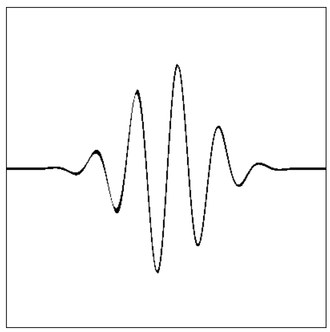
$$d = \frac{|y_1 - mx_1 - b|}{\sqrt{m^2 + 1}} \tag{4}$$

Substitute the corresponding values of m and b from (3) into (4) to obtain

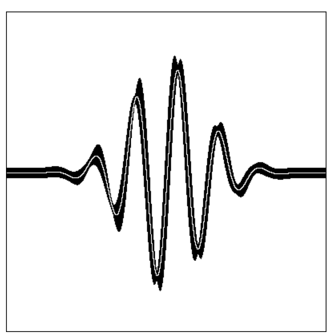
$$\begin{aligned} d &= \frac{|y_1 - f'(x_1)\,x_1 - (f(x_1) - f'(x_1)\,x_1)}{\sqrt{[f'(x_1)]^2 + 1}} \\ &= \frac{|y_1 - f'(x_1)\,x_1 - f(x_1) + f'(x_1)\,x_1|}{\sqrt{[f'(x_1)]^2 + 1}} \\ &= \frac{|y_1 - f(x_1)\,|}{\sqrt{[f'(x_1)]^2 + 1}} \end{aligned} \tag{5}$$

Example

Let's try this with our function $f(x) = 2e^{-x^2}\sin(8x)$. The derivative of the function is $f'(x) = 10e^{-x^2}\cos(8x) - 4x^2e^{-x^2}\sin(8x) = 2e^{-x^2}\left(5\cos(8x) - 2x^2\sin(8x)\right)$. Go through each pixel and colour it black if the distance d obtained from (5) is less than, say, 0.02. This gives the following result:



This produces a continuous representation of the curve. Notice slight variations in thickness near the turning points - this is caused by our previous assumption that the tangent line to the curve would result in a reasonable estimate of the distance. Near turning points, whenever the gradient changes sharply, this leads to an error in the distance estimate. For example, if we re-draw with a larger distance limit, e.g. 0.1, the error becomes more pronounced. The true function is indicated by the thin white line.



Instead of simply using a black & white representation of the distance estimate, we can also map the distance to a colour gradient function, which can produce some very nice images - see below.