Shortest distance between a point and a line

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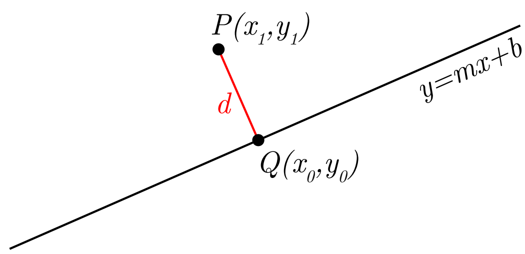
Credit

Various proofs are available on the web of how to calculate the shortest distance between a point and a line, but I found all to be rather unclear, or short on derivation. By far the best I found was a set of course notes by Dr. Will Garner from UCSD, available at http://www.math.ucsd.edu/~wgarner/math4c/. Click on "Derivations" then "Distance between a point and a line". I have presented his proof here, laid out in LaTeX for appearance.

Initial

We have an arbitary line $y = mx + b$ and want to find the shortest distance d between a point $P(x_1, y_1)$ and the line. Let the closest point on the line be $Q(x_0, y_0)$.



Derivation

PQ is perpendicular to $y = mx + b$, so the gradient of PQ is $-\frac{1}{m}$. Therefore, the equation of PQ is

$$y = -\frac{1}{m}x + c \tag{1}$$

The point $P(x_1, y_1)$ lies on PQ. Substitute $x = x_1, y = y_1$ into (1) to obtain

$$c = y_1 + \frac{1}{m}x_1 \tag{2}$$

Substituting this back into (1), we obtain

$$y = -\frac{1}{m}x + \left(y_1 + \frac{1}{m}x_1\right) \tag{3}$$

At the intersection point $Q(x_0, y_0)$, both lines have the same value of y. Therefore, we can set (3) equal to $mx + b$ and solve to find x_0 as follows.

$$mx_0 + b = -\frac{1}{m}x_0 + \left(y_1 + \frac{1}{m}x_1\right)$$

$$
\begin{aligned}
mx_0 + \frac{1}{m}x_0 &= y_1 + \frac{1}{m}x_1 - b \\
\frac{m^2+1}{m}x_0 &= y_1 + \frac{1}{m}x_1 - b \\
x_0 &= \frac{m}{m^2+1}\left(y_1 + \frac{1}{m}x_1 - b\right) \\
x_0 &= \frac{my_1 + x_1 - mb}{m^2+1} \qquad (4)
\end{aligned}
$$

We can find y_0 by substituting (4) into $y = mx + b$.

$$
\begin{aligned}
y_0 &= m\left(\frac{my_1 + x_1 - mb}{m^2+1}\right) + b \\
&= \frac{m^2y_1 + mx_1 - m^2b}{m^2+1} + b \\
&= \frac{m^2y_1 + mx_1 - m^2b + b\left(m^2+1\right)}{m^2+1} \\
&= \frac{m^2y_1 + mx_1 - m^2b + m^2b + b}{m^2+1} \\
&= \frac{m^2y_1 + mx_1 + b}{m^2+1} \qquad (5)
\end{aligned}
$$

Finally, we can find the distance d by using both (4) and (5).

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{\left[x_1 - \left(\frac{my_1 + x_1 - mb}{m^2+1}\right)\right]^2 + \left[y_1 - \left(\frac{m^2y_1 + mx_1 + b}{m^2+1}\right)\right]^2} \\
&= \sqrt{\left[\frac{x_1\left(m^2+1\right) - my_1 - x_1 + mb}{m^2+1}\right]^2 + \left[\frac{y_1\left(m^2+1\right) - m^2y_1 - mx_1 - b}{m^2+1}\right]^2} \\
&= \sqrt{\left[\frac{m^2x_1 + x_1 - my_1 - x_1 + mb}{m^2+1}\right]^2 + \left[\frac{m^2y_1 + y_1 - m^2y_1 - mx_1 - b}{m^2+1}\right]^2} \\
&= \sqrt{\left[\frac{m^2x_1 - my_1 + mb}{m^2+1}\right]^2 + \left[\frac{y_1 - mx_1 - b}{m^2+1}\right]^2} \\
&= \sqrt{\left[\frac{-m\left(y_1 - mx_1 - b\right)}{m^2+1}\right]^2 + \left[\frac{y_1 - mx_1 - b}{m^2+1}\right]^2} \\
&= \sqrt{m^2\frac{\left(y_1 - mx_1 - b\right)^2}{\left(m^2+1\right)^2} + \frac{\left(y_1 - mx_1 - b\right)^2}{\left(m^2+1\right)^2}} \\
&= \sqrt{\frac{m^2+1}{\left(m^2+1\right)^2}\left(y_1 - mx_1 - b\right)^2} \\
&= \sqrt{\frac{\left(y_1 - mx_1 - b\right)^2}{m^2+1}} \\
&= \frac{|y_1 - mx_1 - b|}{\sqrt{m^2+1}} \qquad (6)
\end{aligned}
$$

The last line is obtained from the fact that $\sqrt{x^2} = |x|$ (the root of the square of something is equivalent to the absolute value of the something).